

Existence of the global solutions of an integro-differential equation in population dynamics

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Abstract

We study a nonlinear integro-differential equation arising in population dynamics. It has been already proved by Rybka, Tang and Waxman that it has a unique local in time solution. Here, after deriving appropriate a priori estimates we show that the dynamics is global in time.

Introduction

We study solutions to an integro-differential equation (1), arising in population dynamics, see.^{RTW} The specific problem, we have in mind is set up to model the evolution of an asexual population. Let us describe the basic premise. The organisms are born mature and mutation occurs at the birth. The selection at time $t > 0$ depends only on a single trait x , which is assumed to be a real number. We study evolution of a probability density $\phi(x, t)$ of the trait in the population, taking value x at time $t > 0$. This line of theoretical research in biology was initiated by Kimura.^{Ki} The actual set of equation, we study here, was derived in^{WP1} under the premise that the environment changes at a

constant rate c . The model equation is as follows,

$$\left\{ \begin{array}{l} \frac{\partial \phi}{\partial t} = ((1 - U)\bar{d}(t) - d(x - ct))\phi + U\bar{d}(t)f \star \phi, \\ \phi(x, 0) = \phi_0(x), \\ \phi(x, t) \geq 0 \text{ for all } (x, t) \in (-\infty, \infty) \times (0, \infty), \\ \int_{-\infty}^{\infty} \phi(x, t) dx = 1 \text{ for all } t \geq 0. \end{array} \right. \quad (1)$$

Here U is a positive coefficient, which is less than one, c is a given positive number. The other ingredients will be explained below. This initial value problem has been studied in a series of papers,,^{WP2},^{TW},^{RTW}.^{Sa} In^{WP2} a discrete time version was studied and numerical experiments were performed suggesting existence of a traveling wave. This is a special but very important form of a solution. Such a solution tells us that the organism may follow the changes of the environment. It also gave insight into the shape of such moving distribution density. In^{BTW} the multi-loci version of (1) for a traveling wave was numerically studied. Later, Becker in her thesis^{Be} considered a model of population with a two coupled traits. Since this kind of nonlinear nonlocal equation seems to be new, there has been a lot of open theoretical questions. In^{TW} the authors rigorously established existence of a traveling wave. These results were subsequently improved by.^{Sa} Well-posedness of system (1) was studied in,^{RTW} where unique local-in-time solutions were constructed and their continuous dependence upon the data was shown too. Also the well-posedness of the system linearized around the traveling wave was studied. However, two major questions: (1) global in time existence of solutions; (2) stability of the traveling wave, were left open.

In^{RTW} the authors missed a crucial a priori estimate for the global existence. We derive it here, see Proposition 1 and Theorem 1. However, we impose an additional assumption on the initial datum, namely we require that it has finite first four moments. Our main accomplishment, which is the global existence stated in Theorem 2 is based on that estimate. We stress that this fact is not obvious, since we deal with a nonlinear problem whose growth in ϕ is quadratic.

Let us describe (1) and its components in details. The equation is so normalized that $x = 0$ is the optimal genotypic value. The mortality rate given by

$$d(x) = 1 + x^2,$$

reflects this normalization. Since by assumption the optimal value changes with the velocity c , the mean mortality obtained by averaging d against ϕ is,

$$\bar{d}(t) = \int_{\mathbb{R}} d(x - ct) \phi(x, t) dx. \quad (2)$$

This suggests that ϕ should have at least first two moments. For any probability distribution u its n -th moment is denoted by $M_n(u)$,

$$M_n(u) = \int_{\mathbb{R}} x^n u(x) dx. \quad (3)$$

In our equation, (1) f is the normal distribution with mean m and standard deviation σ , i.e

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

This choice is biologically well-motivated, but our analysis holds for any smooth probability density with finite sufficiently high moments.

We recall that $f \star \phi$ denotes the convolution of f and ϕ , i.e.

$$f \star \phi(x, t) = \int_{\mathbb{R}} f(x - y) \phi(y, t) dy.$$

After describing all the ingredients of (1) we present the plan of our paper. We observe that we have to show enough a priori estimates to prove global in time existence. We mentioned that the density ϕ must have first two moments. In order to guarantee that we found that it is advantageous to work with the following Banach space,

$$X = \{u \in C^0(\mathbb{R}) : \sum_{j=0}^2 p_{2j}(u) < \infty\}, \quad \text{where } p_j(u) = \|x^j u(x)\|_{L^\infty(\mathbb{R})}.$$

In^{RTW} it was shown, that for a positive $T > 0$ equation (1) system has a unique solution $\phi \in C^1([0, T), X)$.

It turns out however that, an additional assumption of finiteness of the fourth moment of initial data guarantees that the solution is global in time. This is the content of Theorem 2. The first step in the proof is an observation that the second moment of the solution defined on interval $(0, T)$ is time integrable, provided that T is finite. This in turn implies that the fourth moment is uniformly bounded over $(0, T)$. As a result, we may deduce that the solution $\phi(\cdot, t)$ is not only bounded in X , but also it forms a Cauchy sequence if $t_k \rightarrow T$. This analysis depends in an essential way on the constant variation formula for (1).

Estimate of moments

Experience gained in^{RTW} suggests that it is easier to work with the integral form of (1). The constant variation formula applied to (1) yields,

$$\begin{aligned} \phi(x, t) = & \exp\left(-\int_0^t [d(x - cs) - (1 - U)\bar{d}(s)] ds\right) \phi_0(x) \\ & + \int_0^t \exp\left(-\int_s^t [d(x - c\tau) - (1 - U)\bar{d}(\tau)] d\tau\right) U\bar{d}(s) f \star \phi(x, s) ds. \end{aligned} \quad (4)$$

Equation (4) is indeed nonlinear, for \bar{d} depends upon ϕ . However, for a fixed \bar{d} (4) becomes a linear equation for ϕ . We will use this property to squeeze out the first estimate below.

Proposition 1. *If ϕ is a unique solution to (1), defined on $[0, T)$, then function $\bar{d}(s)$ belongs to $L^1(0, T)$.*

Proof. From now on, we will work with the integral equation (4). Integrating both sides of (4)

with respect to x over \mathbb{R} , we get

$$\begin{aligned}
1 &= \int_{\mathbb{R}} \phi(x, t) dx \\
&= \int_{\mathbb{R}} \left[\exp \left(- \int_0^t [d(x - cs) - (1 - U)\bar{d}(s)] ds \right) \phi_0(x) \right. \\
&\quad \left. + \int_0^t \exp \left(- \int_s^t [d(x - c\tau) - (1 - U)\bar{d}(\tau)] d\tau \right) U \bar{d}(s) f \star \phi(x, s) ds \right] dx.
\end{aligned}$$

Since both terms on the right-hand-side (RHS) are non-negative, we conclude that

$$1 \geq \exp \left((1 - U) \int_0^t \bar{d}(s) ds \right) \int_{\mathbb{R}} \exp \left(- \int_0^t d(x - cs) ds \right) \phi_0(x) dx. \quad (5)$$

In order to extract the desired estimate, we examine the following expression in detail,

$$\int_{\mathbb{R}} \exp \left(- \int_0^t d(x - cs) ds \right) \phi_0(x) dx = e^{-t - c^2 \frac{t^3}{3}} \int_{\mathbb{R}} e^{-x^2 t + cxt^2} \phi_0(x) dx = e^{-t - c^2 \frac{t^3}{3}} J.$$

We choose $R = R_{\phi_0}$ such that

$$\int_{|x| < R} \phi_0(x) = \frac{1}{2}. \quad (6)$$

If we take into account that

$$\inf_{\{x \in \mathbb{R}: |x| < R\}} \exp \left(-x^2 t + cxt^2 \right) = \exp \left(-R^2 t - cRt^2 \right),$$

then we have the following bound on J

$$J \geq \int_{|x| < R} e^{-x^2 t + cxt^2} \phi_0(x) dx \geq e^{-R^2 t - cRt^2} \int_{|x| < R} \phi_0(x) dx = \frac{1}{2} e^{-R^2 t - cRt^2}.$$

The last inequality holds because of (6). Keeping this in mind, we can deduce that (5) implies

the following inequality,

$$C(t, \phi_0) := 2 \exp \left(t + c^2 \frac{t^3}{3} \right) \exp \left(R^2 t + c R t^2 \right) \geq \exp \left((1 - U) \int_0^t \bar{d}(s) ds \right). \quad (7)$$

Since $t \rightarrow C(t, \phi_0)$ is increasing with respect to t , $t \leq T < \infty$ and \bar{d} is positive we infer that

$$\int_0^T \bar{d}(s) ds \leq \frac{\ln C(T, \phi_0)}{1 - U} < \infty.$$

□

Subsequently, we will use simple estimates of probability density moments.

Lemma 1. *For any probability measure f , the following inequalities hold*

$$|M_n(f)| \leq 1 + M_{n+2}(f) \text{ when } n \text{ is even,}$$

$$|M_n(f)| \leq 1 + M_{n+1}(f) \text{ when } n \text{ is odd.}$$

Here, M_n is the n -th moment defined by (3).

We leave an easy proof to the reader.

In particular, this lemma implies the following fact.

Corollary 1. *If f is any probability density, then for any $1 \leq i \leq 4$, we have*

$$|M_i(f)| \leq 2 + M_4(f).$$

Proof. Indeed, Lemma 1 gives us

$$|M_1(f)| \leq 1 + M_2(f) \leq 2 + M_4(f), \quad |M_3(f)| \leq 1 + M_4(f). \quad (8)$$

□

Now, we shall show propagation of regularity understood as finiteness of moments.

Theorem 1. *Let us suppose that the probability distribution $\phi(\cdot, t)$ is the solution of (4) and ϕ_0 has finite first four moments. Then,*

(a) *for all $t \in [0, T]$ solution $\phi(\cdot, t)$ has finite first four moments; moreover,*

$$\sup_{t \in [0, T]} M_n(\phi(\cdot, t)) < \infty. \quad n = 1, 2, 3, 4.$$

(b)

$$\sup_{t \in [0, T]} \|\phi(\cdot, t)\|_X < M(T) < \infty.$$

Proof. For the sake of simplicity of notation we will write $M_n(t)$ in place of $M_n(\phi(\cdot, t))$. We notice that equation (4), positivity of \bar{d} and estimate (7) imply that

$$|M_n(t)| \leq C(T, \phi_0)|M_n(0)| + C(T, \phi_0) \int_0^t \bar{d}(s) \int_{\mathbb{R}} x^n (f \star \phi)(x, s) dx ds. \quad (9)$$

We want to express the integrand of the RHS of (9) in terms of $M_n(s)$. For this purpose, we have to transform x^n into a more convenient form. We recall that Newton binomial formula yields

$$x^n = (x - y)^n + \sum_{k=1}^n \binom{n}{k} (x - y)^{n-k} y^k.$$

If we apply this formula to the integral on the RHS of (9), then we shall see it may be estimated by the following expression

$$\begin{aligned} & \left| \int_0^t \bar{d}(s) \int_{\mathbb{R}} \int_{\mathbb{R}} (x - y)^n f(y) \phi(x - y, s) dy dx ds \right| \\ + & \left| \sum_{k=1}^n \binom{n}{k} \int_0^t \bar{d}(s) \int_{\mathbb{R}} \int_{\mathbb{R}} (x - y)^{n-k} y^k f(y) \phi(x - y, s) dy dx ds \right| =: Z_1 + Z_2. \end{aligned}$$

If we substitute $z = x - y$ and keep in mind the definition of f , then we notice that

$$Z_1 + Z_2 = \int_0^t \bar{d}(s) M_n(s) ds + \sum_{k=1}^n \binom{n}{k} \int_0^t \bar{d}(s) |M_k(f)| |M_{n-k}(s)| ds$$

Now, we invoke Corollary 1, and take $n = 4$, then for any $k = 1, 2, 3$ we have

$$|M_{n-k}(s)| \leq 2 + M_4(s).$$

If we write C_f for $\max_{k \in \{1, \dots, n\}} |M_k(f)|$, then (9) becomes

$$M_4(s) \leq C(T, \phi_0) M_4(0) + 16C_f C(T, \phi_0) \int_0^t \bar{d}(s) M_4(s) ds.$$

Since $\bar{d}(s)$ is integrable, Gronwall inequality implies

$$M_4(t) \leq M_4(0) C(T, \phi_0) \exp \left(16C_f C(T, \phi_0) \int_0^t \bar{d}(s) ds \right), \quad 0 \leq t \leq T.$$

Combining this inequality with Corollary 1 yields the claim.

We now show part (b). By the integral equation and the definition of p_0 , we notice:

$$\begin{aligned} p_0(\phi(\cdot, t)) &= \|\phi(x, t)\|_{L^\infty(\mathbb{R})} \\ &\leq \left\| \exp \left(\int_0^t (1 - U) \bar{d}(s) ds \right) \phi_0(x) \right\|_{L^\infty(\mathbb{R})} \\ &\quad + \left\| \int_0^t \exp \left(- \int_s^t [d(x - c\tau) - (1 - U) \bar{d}(\tau)] d\tau \right) U \bar{d}(s) f \star \phi(x, s) ds \right\|_{L^\infty(\mathbb{R})} \\ &\leq C(T, \phi_0) \|\phi_0\|_{L^\infty(\mathbb{R})} + \frac{C(T, \phi_0) U}{\sqrt{2\pi\sigma}} \int_0^T |\bar{d}(s)| ds =: P_0 < \infty. \end{aligned} \tag{10}$$

We used in these calculations the following bound

$$\|f \star \phi\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{R})} = \frac{1}{\sqrt{2\pi\sigma}}.$$

We can also see that

$$\begin{aligned}
p_2(\phi(\cdot, t)) &= \|x^2\phi(x, t)\|_{L^\infty(\mathbb{R})} \\
&\leq C(T, \phi_0) \|x^2\phi_0\|_{L^\infty(\mathbb{R})} + C(T, \phi_0)U \left\| \int_0^T x^2 \bar{d}(s) f \star \phi(x, s) ds \right\|_{L^\infty(\mathbb{R})} \\
&\leq C(T, \phi_0)p_2(\phi_0) + C(T, \phi_0)U \int_0^T |\bar{d}(s)| \|x^2 f \star \phi\|_{L^\infty(\mathbb{R})} ds.
\end{aligned}$$

In order to estimate the expression above, one can look at $\|x^2 f \star \phi\|_{L^\infty(\mathbb{R})}$. Using an elementary inequality $x^2 \leq 2y^2 + 2(x - y)^2$, we get immediately

$$\begin{aligned}
\|x^2 f \star \phi(\cdot, t)\|_{L^\infty(\mathbb{R})} &\leq 2 \int_{\mathbb{R}} f(x - y) y^2 \phi(y, t) dy + 2 \int_{\mathbb{R}} \phi(y, t) f(x - y) (x - y)^2 dy \\
&\leq \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \int_{\mathbb{R}} y^2 \phi(y, t) dy + 2p_2(f) \int_{\mathbb{R}} \phi(y, t) dy = \frac{\sqrt{2}}{\sqrt{\pi}\sigma} M_2(\phi(\cdot, t)) + 2p_2(f).
\end{aligned}$$

Thus,

$$\sup_{t \in [0, T]} p_2(\phi(\cdot, t)) \leq C(T, \phi_0)p_2(\phi_0) + C(T, \phi_0)U \int_0^T \bar{d}(s) ds \frac{\sqrt{2}}{\sqrt{\pi}\sigma} \sup_{t \in [0, T]} M_2(\phi) + 2p_2(f) =: P_2 < \infty \quad (11)$$

Now, we turn our attention to $p_4(\phi(\cdot, t))$.

$$\begin{aligned}
p_4(\phi(\cdot, t)) &= \|x^4\phi(x, t)\|_{L^\infty(\mathbb{R})} \\
&\leq C(T, \phi_0) \|x^4\phi_0(x)\|_{L^\infty(\mathbb{R})} + C(T, \phi_0)U \left\| \int_0^T x^4 \bar{d}(s) f \star \phi(x, s) ds \right\|_{L^\infty(\mathbb{R})} \\
&\leq p_4(\phi_0) + \int_0^T |\bar{d}(s)| \|x^4 f \star \phi(x, s)\|_{L^\infty(\mathbb{R})} ds.
\end{aligned}$$

In order to estimate this, we look at $\|x^4 f \star \phi(x, s)\|_{L^\infty(\mathbb{R})}$. We notice that for all real x, y the following inequality holds:

$$x^4 \leq 5y^4 + 5(x - y)^4 + 6y^2(x - y)^2.$$

Using this inequality, we continue our calculations,

$$\begin{aligned}
\|x^4 f \star \phi\|_{L^\infty(\mathbb{R})} &\leq 5 \int_{\mathbb{R}} f(x-y) y^4 \phi(y, s) dy \\
&\quad + 6 \int_{\mathbb{R}} \phi(y, s) f(x-y) y^2 (x-y)^2 dy + 5 \int_{\mathbb{R}} \phi(y, s) f(x-y) (x-y)^4 dy \\
&\leq \frac{5}{\sqrt{2\pi}\sigma} M_4(\phi) + 6M_2(\phi)p_2(f) + 5p_4(f).
\end{aligned}$$

Therefore,

$$\sup_{t \in [0, T)} p_4(\phi(\cdot, t)) \leq C(T, \phi_0) p_4(\phi_0) + C(T, \phi_0) U C_4 \int_0^T |\bar{d}(s)| ds =: P_4. \quad (12)$$

Now, we can take for $M = \max\{P_0, P_2, P_4\}$. □

Existence of global solutions

In this section we show our main result, the global in time solution to (1). It is based upon the estimates established in Section 2. They use in an essential way the additional regularity of the initial condition, i.e. the finite fourth moment. These bounds depend upon time, but for each time constant t they are finite.

Theorem 2. *Let us assume that $\phi_0 \in X$ and $M_4(\phi_0) < \infty$, then the unique solution to (1) exists for all $t \in [0, \infty)$, moreover $\phi \in C([0, \infty), X)$.*

Proof. We have already constructed a unique solution $\phi \in C([0, T), X)$. If $T < \infty$, then we will show, that $\lim_{t \rightarrow T} \phi(t)$ exists in X . Hence, due to Theorem 1 $\phi(T)$ automatically has a finite fourth moment, $M_4\phi(T)$. Subsequently, ϕ may be extended to $[T, T + \epsilon)$ for a positive ϵ . Hence our claim will follow.

Let us suppose that $\{t_m\}$ is any sequence converging to T . We shall see, that $\{\phi(t_n)\}$ is a Cauchy sequence in the Banach space X . In order to achieve this goal we will use the constant

variation formula. For a given $\epsilon > 0$ we shall estimate the norm of the difference $\phi_n - \phi_m$,

$$\|\phi_n - \phi_m\|_X = \|\phi(x, t_n) - \phi(x, t_m)\|_{L^\infty(\mathbb{R})} + \|x^2(\phi(x, t_n) - \phi(x, t_m))\|_{L^\infty(\mathbb{R})} + \|x^4(\phi(x, t_n) - \phi(x, t_m))\|_{L^\infty(\mathbb{R})}$$

In order to simplify our subsequent calculations, we introduce the following notation

$$g(x, s, t_n) = \int_s^{t_n} [d(x - c\tau) - (1 - U)\bar{d}(\tau)] d\tau. \quad (13)$$

The integral equation (4) implies that for $j \in \{0, 2, 4\}$, we have

$$|x^j(\phi_n - \phi_m)(x, \cdot)| \leq I_{j1}(x) + I_{j2}(x),$$

where

$$I_{j1}(x) = \left| e^{-g(x, 0, t_n)} (1 - e^{-g(x, t_n, t_m)}) x^j \phi_0(x) \right|,$$

$$I_{j2}(x) = \left| \int_0^{t_n} e^{-g(x, s, t_n)} (1 - e^{-g(x, t_n, t_m)}) U \bar{d}(s) x^j f \star \phi(x, s) + \int_{t_n}^{t_m} e^{-g(x, s, t_m)} U \bar{d}(s) x^j f \star \phi(s, x) \right| ds.$$

In the following calculations, without loss of generality, we may assume that $t_n < t_m$. First, we

shall take care of $I_{j1}(x)$, $j \in \{0, 2, 4\}$. Let us notice first that $\sup_{x \in \mathbb{R}} I_{j1}(x) = \max\{\sup_{|x| \leq R} I_{j1}(x), \sup_{|x| > R} I_{j1}(x)\}$.

Subsequently, positivity of d , (13) and (7) imply that

$$\sup_{|x| \leq R} I_{j1}(x) \leq C(\phi_0, T) \sup_{|x| \leq R} \left| (1 - e^{-\int_{t_n}^{t_m} d(x - cs) ds}) \phi_0 x^j \right|.$$

If $R^2 \geq 1 + 2c^2 T^2$, then we have

$$d(x - ct) \leq 1 + 2R^2 + 2c^2 T^2 < 3R^2.$$

Once we fixed R we can see that

$$\sup_{|x| \leq R} I_{j1}(x) \leq C(\phi_0, T) (1 - e^{-3R^2(t_m - t_n)}) \|\phi_0(x)\|_X \leq \epsilon \quad \text{when } t_m - t_n \text{ is small enough.}$$

Now, we turn our attention to the estimate of $\sup_{|x|>R} I_{j1}(x)$. We may assume that R is so large, that for our ϵ the following bound holds

$$C(\phi_0, T)e^{-\int_0^{t_m} d(x-cs) ds} \leq \frac{\epsilon}{\|\phi_0(x)\|_X} \text{ for } |x| > R.$$

Therefore, again due to positivity of d , we have

$$\sup_{|x|>R} I_{j1}(x) \leq C(\phi_0, T) \left| \frac{\epsilon}{\|\phi_0(x)\|_X} (1 - e^{-\int_{t_n}^{t_m} d(x-cs) ds}) \phi_0 x^j \right| \leq \epsilon.$$

The estimates of $I_{j2}(x)$ are performed in similar fashion, but they have to be slightly more subtle. The problem is that $s \in [0, t_n]$ appearing under the integral sign, need not be close to T .

Thus, $I_{j2}(x)$, $j = 0, 2, 4$ take the following form

$$I_{j2}(x) \leq \left| \int_0^{t_n} (e^{-g(x,s,t_n)} - e^{-g(x,s,t_m)}) U \bar{d}(s) x^j f \star \phi(x, s) ds \right| + \left| \int_{t_n}^{t_m} e^{-g(x,s,t_m)} U \bar{d}(s) x^j f \star \phi(x, s) ds \right|.$$

The second term is easy to estimate. For any $j \in \{0, 2, 4\}$ integrability of \bar{d} and (13) imply that

$$\left| \int_{t_n}^{t_m} e^{-g(x,s,t_m)} U \bar{d}(s) x^j f \star \phi(x, s) ds \right| \leq U P_j \int_{t_n}^{t_m} e^{\|d\|_{L^1}} \rightarrow 0, \text{ as } t_n, t_m \rightarrow T,$$

where P_j , $j = 0, 2, 4$, were defined in (10), (11), (12).

In order to bound the first integral, we split the interval $[0, t_n]$ into two $[0, t_n - \delta]$ and $[t_n - \delta, t_n]$, where $\delta > 0$ shall be selected later.

We notice that (7), Theorem 1 and positivity of d yield the following estimate,

$$\begin{aligned} \left| \int_{t_n - \delta}^{t_n} e^{-g(x,s,t_n)} (1 - e^{-g(x,t_n,t_m)}) U \bar{d}(s) x^j f \star \phi(x, s) ds \right| &\leq C(\phi_0, T) U \left| \int_{t_n - \delta}^{t_n} \bar{d}(s) x^j f \star \phi(x, s) ds \right| \leq \\ &\leq C P_j \int_{t_n - \delta}^{t_n} \bar{d}(s) ds < \epsilon. \end{aligned}$$

The last inequality is a result of integrability of \bar{d} implying existence of sufficiently small δ with

the desired property.

Now, we have to estimate the supremum over the real number of the integral over $[0, t_n - \delta]$.

First we notice that

$$\begin{aligned}
& \left| \int_0^{t_n - \delta} e^{-g(x, s, t_n)} (1 - e^{-(g(x, t_n, t_m))}) U \bar{d}(s) x^j f \star \phi(x, s) ds \right| \leq \\
& P_j C(\phi_0, T) \sup_{x \in \mathbb{R}} \left(\int_0^{t_n - \delta} e^{-\int_s^{t_n} d(x - c\tau) d\tau} (1 - e^{-\int_{t_n}^{t_m} d(x - c\tau) d\tau}) \bar{d}(s) ds \right. \\
& \quad \left. + \int_0^{t_n - \delta} e^{-\int_{t_n}^{t_m} d(x - c\tau) d\tau} (1 - e^{\int_{t_n}^{t_m} \bar{d}(\tau) d\tau}) \bar{d}(s) ds \right) \leq \\
& P_j C(\phi_0, T) \left(\sup_{x \in \mathbb{R}} \int_0^{t_n - \delta} e^{-\int_s^{t_n} d(x - c\tau) d\tau} (1 - e^{-\int_{t_n}^{t_m} d(x - c\tau) d\tau}) \bar{d}(s) ds + \int_0^{t_n - \delta} (1 - e^{\int_{t_n}^{t_m} \bar{d}(\tau) d\tau}) \bar{d}(s) ds \right).
\end{aligned}$$

We have already noticed that the second integral can be made smaller than ϵ for large n and m .

In order to estimate the first integral on the RHS above we use again that for any positive function u , $\sup_{x \in \mathbb{R}} u = \max\{\sup_{|x| > R} u, \sup_{|x| \leq R} u\}$. For a given $\epsilon > 0$ and δ fixed above we may possibly take larger $R > 0$ so that,

$$3R^2 \geq d(x - c\tau) \geq \frac{1}{2}R^2 \quad \text{and} \quad e^{-\frac{1}{2}R^2\delta} < \frac{\epsilon}{P_j \|\bar{d}\|_{L^1}}.$$

With this choice of R for $|x| \leq R$ we see that for sufficiently large m, n we have,

$$1 - e^{-\int_{t_n}^{t_m} d(x - c\tau) d\tau} \leq 1 - e^{-\int_{t_n}^{t_m} 3R^2 ds} < \epsilon.$$

We conclude that

$$\sup_{|x| \leq R} \int_0^{t_n - \delta} e^{-\int_s^{t_n} d(x - c\tau) d\tau} (1 - e^{-\int_{t_n}^{t_m} d(x - c\tau) d\tau}) \bar{d}(s) ds$$

can be made as small as we wish. Moreover, we have,

$$\sup_{|x|>R} \int_0^{t_n-\delta} e^{-\int_s^{t_n} d(x-c\tau) d\tau} (1 - e^{-\int_{t_n}^{t_m} d(x-c\tau) d\tau}) \bar{d}(s) ds \leq C \int_0^{t_n-\delta} e^{-\frac{1}{2}R^2\delta} \bar{d}(s) ds < \epsilon.$$

We then conclude that

$$\sup_{x \in \mathbb{R}} I_{j2}(x) \leq 3\epsilon.$$

Subsequently, $\{\phi(t_n)\}$ is a Cauchy sequence, hence the limit $\lim_{t \rightarrow T} \phi(t)$ exists in X and we may invoke the existence result in ^{RTW} to continue the solution onto $[T, T + \epsilon)$. Our claim follows. \square

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